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Determination of the lower natural frequencies of circular plates with mixed boundary conditions

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Abstract

The natural frequencies are obtained for elastic circular plates with mixed boundary conditions. The boundary conditions of the plate were combinations of clamped, simply supported, free, and guided. The natural frequencies are presented as a function of the angle over which the circumference of the plate is treated as one boundary and the remainder as another boundary condition.

It was found that the axisymmetric modes exhibit continuous variation of the natural frequency from one pure condition to the other pure boundary condition, while the asymmetric modes show two branches of varying curvature and magnitude fluctuation, depending on the magnitude of the angles of mixed boundaries.

1. Introduction

In many cases, some of which may be accidental, the boundary conditions of an elastic plate are altered by fracture along a part of its periphery into a different case, thus producing mixed boundary conditions. To determine the influence of this new mixed boundary condition and how the magnitude of such a fracture alters the natural frequencies of the circular plate, an approximate method has been developed to find the lower natural frequencies.

The problem of bending of a circular plate with mixed boundaries has been treated previously for some special conditions. A more general problem for a plate partly clamped and partly simply supported was formulated in Ref. [1], which treats forced vibrations due to a harmonic load perpendicular to the plate, which in addition is subjected to a compressive load. In later papers [2–4] a variational approach was applied to a circular plate partially clamped and partially simply supported. One method is based upon two perturbations, i.e. one when the plate is clamped all around, the other when the plate is simply supported. The first perturbation yielded upper bounds for the eigenvalues, while the latter presented the lower bounds. There are, however, still some discrepancies with previously published results, all of which are restricted only to the mixed boundary conditions of a partly clamped and partly simply supported system.

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Nomenclature

a	radius of circular plate
D	Stiffness of plate ($D = Eh^3/(12(1-\nu^2))$)
E	modulus of elasticity
h	thickness of plate
I	moment of inertia
I_m	modified Bessel function
J_m	Bessel function
k	distributed stiffness of translational springs (force/unit length)
K	distributed stiffness of spiral springs (moment/unit length)
M	bending moment, twisting moment
N_1, N_2	number of points, at which the boundary conditions are satisfied
Q	transverse shearing force
r, φ	polar coordinates
t	time
V	Kelvin–Kirchhoff edge reaction
$w(r, \varphi, t)$	displacement of plate
α	angle
ϱ	mass density of plate
ν	Poisson's ratio
λ	eigenvalue

With the advent of very efficient high speed computers, allowing solution of a large number of algebraic equations in a relatively short time, we have proposed another method for determining the eigenvalues under various mixed boundary conditions [5], where not only the fundamental natural frequency is determined, but also higher axi- and asymmetric mode shapes of the plate are investigated. In addition we have determined the nodal lines of the asymmetric natural modes.

It should be mentioned that the method presented is not restricted to the determination of eigenfrequencies of unloaded plates, but may also be applied to radially loaded plates. It also allows the investigation of the response of the plate to harmonically oscillating loads perpendicular to the plate as well as the buckling of circular plates with mixed boundaries. In addition, the method may also be used for rectangular plates with mixed boundary conditions, requiring treatments in Cartesian coordinates.

In Ref. [5] we treated a plate, part of which was clamped, and the remainder of whose boundary was considered to be simply supported. A new phenomenon was detected: for the asymmetric modes (angular mode number $m \neq 0$) two natural frequencies and two nodal lines exist. These appear only for mixed boundaries and disappear as soon as the boundary of the plate is either totally clamped or totally simply supported. If α is the magnitude of the angular region of one boundary condition and $(2\pi - \alpha)$ that of the remaining region of the other boundary condition, the natural frequencies exhibit for $\alpha = 0$ the natural frequencies of the pure boundary condition, while $\alpha = 2\pi$ yields those of the other completely pure boundary, as is indicated in the numerical results in the following figures.

In the following we shall investigate the lower natural frequencies of a circular plate for which mixed boundary conditions, such as clamped, simply supported, free or guided conditions are present.

2. Basic equations

The problem of finding the approximate lower natural frequencies of a circular plate exhibiting partially mixed boundary conditions along its periphery may be solved with a semi-analytical method as shown below. This method satisfying the various boundary conditions for a finite number of points at the periphery $r = a$ may be applied to a large variety of boundary conditions. We shall treat here the boundary conditions of clamped, simply supported, free, guided and elastically supported edges of various peripheral edge ranges.

The basic equations require the solution of the equation of the circular plate

$$D\nabla^4 w + \varrho h \frac{\partial^2 w}{\partial t^2} = 0, \qquad (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

with the appropriate mixed boundary conditions. The bending and twisting moments are given by

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)\right], \tag{2}$$

$$M_\varphi = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + v\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right], \tag{3}$$

$$M_{r\varphi} = -D(1-v)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \varphi}\right) \tag{4}$$

and the transverse shearing forces are

$$Q_r = -D\frac{\partial}{\partial r}(\nabla^2 w), \tag{5}$$

$$Q_\varphi = -D\frac{\partial}{r\partial \varphi}(\nabla^2 w), \tag{6}$$

while the Kelvin–Kirchhoff edge reactions are given by

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\varphi}}{\partial \varphi}, \tag{7}$$

$$V_\varphi = Q_\varphi + \frac{\partial M_{r\varphi}}{\partial r}. \tag{8}$$

The displacement of the plate is $w(r,\varphi,t)$; ϱ is its density, h its thickness and $D = Eh^3/(12(1-v^2))$ its bending stiffness; v is Poisson's ratio. The boundary conditions may be either

$$\begin{aligned}
&\text{(a) clamped:} && w = 0 \quad \text{and} \quad \frac{\partial w}{\partial r} = 0 \ \text{at } r = a,\\
&\text{(b) simply supported:} && w = 0 \quad \text{and} \quad M_r = 0 \ \text{at } r = a,\\
&\text{(c) free:} && M_r = 0 \quad \text{and} \quad V_r = 0 \ \text{at } r = a,\\
&\text{(d) guided:} && \frac{\partial w}{\partial r} = 0 \quad \text{and} \quad V_r = 0 \ \text{at } r = a,\\
&\text{(e) elastically supported:} && M_r - K\frac{\partial w}{\partial r} = 0 \quad \text{and} \quad V_r + kw = 0 \ \text{at } r = a,
\end{aligned} \tag{9}$$

where K is the distributed stiffness, i.e. moment/unit length, opposing the edge rotation, and where k is the distributed stiffness, i.e. force/unit length, opposing the translational motion in direction w.

The solution of Eq. (1) yields with $w(r,\varphi,t) = W(r,\varphi)e^{i\omega t}$ the expression

$$W(r,\varphi) = \sum_{m=0}^{\infty}\left\{A_m J_m\left(\lambda\frac{r}{a}\right) + C_m I_m\left(\lambda\frac{r}{a}\right)\right\}\left\{\begin{matrix}\cos m\varphi\\ \sin m\varphi\end{matrix}\right\}, \tag{10}$$

where $\lambda^2 = \omega a^2\sqrt{\varrho h/D}$. The problem of finding the approximate lower natural frequencies for mixed edge conditions depends on the range of the various boundary conditions at hand. The complexity of the numerical procedure is reduced, if two of the four boundary conditions exhibit the same equation. This means that only a set of two infinite equations have to be solved. This is observed for a combination of some of the following boundary condition cases.

The mixed boundary conditions clamped–simply supported exhibit at $r = a$ the same boundary condition $w = 0$ at $r = a$ as does the case for a totally clamped plate and that for a totally simply supported plate. For a clamped–guided mixed boundary case the total range $0\leqslant\varphi<2\pi$ shows $\partial w/\partial r=0$ at $r=a$. For a combination of simply supported and free boundaries, the total range $0\leqslant\varphi<2\pi$ has to satisfy the vanishing bending moment $M_r=0$ at $r=a$, while for a combination of free and guided boundaries the Kelvin–Kirchhoff edge reaction $V_r=0$ at $r=a$, i.e.

$$V_r=-D\left[\frac{\partial}{\partial r}(\nabla^2 w)+\frac{(1-\nu)}{r}\frac{\partial^2}{\partial r\partial\varphi}\left(\frac{1}{r}\frac{\partial w}{\partial\varphi}\right)\right]=0,$$

or

$$V_r=-D\left[\frac{\partial^3 w}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^2}\frac{\partial w}{\partial r}+\frac{(2-\nu)}{r^2}\frac{\partial^3 w}{\partial r\partial\varphi^2}-\frac{(3-\nu)}{r^3}\frac{\partial^2 w}{\partial\varphi^2}\right]=0. \tag{11}$$

It may be mentioned that for a "pure" boundary condition, i.e. with $\alpha=0$, the values λ^2 always yield the well-known natural frequencies of that particular boundary condition case are obtained, while for the "pure" boundary condition of the other case, i.e. $\alpha=2\pi$, the values λ^2 of that particular boundary condition case are obtained. For mixed boundary condition cases of varying $\alpha\neq 0$ and $\alpha\neq 2\pi$ the approximate λ^2-value is obtained from the following procedure, and requires for different modes m and n careful selection of the number of points at which the remaining boundary conditions have to be satisfied to yield acceptable final results. This point is addressed in the numerical solutions below.

3. Method of solution

Let us first treat those cases requiring only two different boundary conditions for the numerical procedure. Then two of the four boundary conditions are valid for the entire boundary range for the two mixed boundaries under consideration. This fact reduces the complexity of the numerical treatment considerably.

There are four basic boundary cases describing such plate oscillations, which we shall treat in the following.

3.1. Clamped along part of the boundary and simply supported along the remainder

If the plate is clamped in the range $0<\varphi<\alpha$ and simply supported in the range $\alpha\leqslant\varphi\leqslant 2\pi$, the plate exhibits for the total range $0\leqslant\varphi<2\pi$ a vanishing deflection $w=0$ at $r=a$ (Fig. 1) and yields therefore for the solution

$$W(r,\varphi)=\sum_{m=0}^{\infty}\left[J_m\left(\lambda\frac{r}{a}\right)-\frac{J_m(\lambda)}{I_m(\lambda)}I_m\left(\lambda\frac{r}{a}\right)\right]\{A_m\cos m\varphi+B_m\sin m\varphi\}. \tag{12}$$

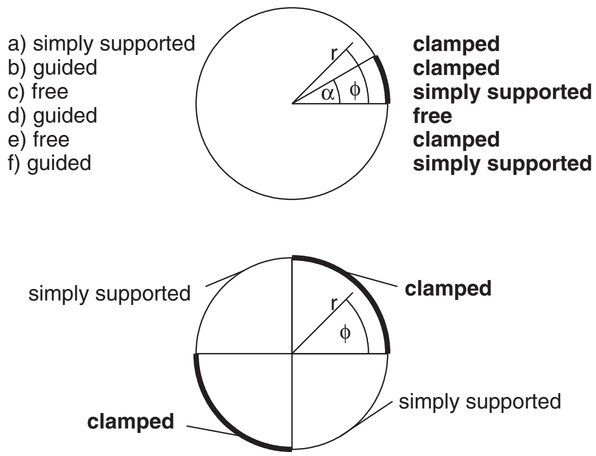

Fig. 1. Geometry and boundaries of the plate system.

Table 1
Eigenvalues $\lambda_{mn}^{(c)2}$ for a clamped plate

$n\backslash m$	0	1	2	3
1	10.2158	21.2604	34.8770	51.0300
2	39.7711	60.8287	84.5826	111.0214
3	89.1041	120.0792	153.8151	190.3038
4	158.1842	199.0534	242.7206	289.1799

Table 2
Eigenvalues $\lambda_{mn}^{(ss)2}$ for a simply supported plate ($\nu = 0.3$)

$n\backslash m$	0	1	2	3
1	4.9351	13.8982	25.6133	39.9573
2	29.7200	48.4789	70.1170	94.5490
3	74.1561	102.7733	134.2978	168.6749
4	138.3181	176.8012	218.2026	262.4847

It satisfies $W(a, \varphi) = 0$ in the total φ-region. If $\alpha = 2\pi$, i.e. the plate is totally clamped the second boundary condition $\partial w/\partial r = 0$ at $r = a$ yields

$$J'_m(\lambda) - \frac{J_m(\lambda)}{I_m(\lambda)} I'_m(\lambda) = 0, \tag{13}$$

which yields the values $\lambda_{mn}^{(c)2}$ given in Ref. [6] and Table 1.

If the plate is totally simply supported, the values $\alpha = 0$ and Eq. (12) has to satisfy in addition the vanishing bending moment $M_r = 0$ at the edge $r = a$. This yields with Eq. (2)

$$\left[J''_m(\lambda) + \frac{\nu}{\lambda} J'_m(\lambda)\right] I_m(\lambda) - \left[I''_m(\lambda) + \frac{\nu}{\lambda} I'_m(\lambda)\right] J_m(\lambda) = 0, \tag{14}$$

which solution results for $\nu = 0.3$ in the eigenvalues $\lambda_{mn}^{(ss)2}$ given in Ref. [6] and Table 2.

For a plate of partly clamped boundary in the range $0 < \varphi < \alpha$ and a simply supported boundary in the range $\alpha \leqslant \varphi \leqslant 2\pi$ we have to satisfy $\partial w/\partial r = 0$ at $r = a$ resulting in

$$\sum_{m=0}^{\infty} \left[J'_m(\lambda) - \frac{J_m(\lambda)}{I_m(\lambda)} I'_m(\lambda)\right] (A_m \cos m\varphi + B_m \sin m\varphi) = 0 \quad \text{in the range } 0 < \varphi < \alpha \tag{15}$$

and $M_r = 0$ in the range $\alpha \leqslant \varphi \leqslant 2\pi$ resulting in

$$\sum_{m=0}^{\infty} \left\{ \left[J''_m(\lambda) + \frac{\nu}{\lambda} J'_m(\lambda)\right] - \frac{J_m(\lambda)}{I_m(\lambda)} \left[I''_m(\lambda) + \frac{\nu}{\lambda} I'_m(\lambda)\right] \right\} (A_m \cos m\varphi + B_m \sin m\varphi) = 0$$
$$\text{in the range } \alpha \leqslant \varphi \leqslant 2\pi. \tag{16}$$

Eqs. (15) and (16) have to be satisfied at a chosen number of points in each range. If $\varphi = \alpha n_1/(N_1 + 1)$ with $n_1 = 1, 2, \ldots, N_1$ in the range $0 < \varphi < \alpha$ and $\varphi = \alpha + (2\pi - \alpha) n_2/N_2$ with $n_2 = 0, 1, \ldots, N_2$ in the range $\alpha \leqslant \varphi \leqslant 2\pi$, Eqs. (15) and (16) read then ($N_1 + N_2$ even)

$$\sum_{m=0}^{(N_1+N_2)/2} \left[J'_m(\lambda) - \frac{J_m(\lambda)}{I_m(\lambda)} I'_m(\lambda)\right] \left\{ A_m \cos\left(\frac{m\alpha n_1}{N_1 + 1}\right) + B_m \sin\left(\frac{m\alpha n_1}{N_1 + 1}\right) \right\} = 0$$
$$\text{for } n_1 = 1, 2, \ldots, N_1 \tag{17}$$

and

$$\sum_{m=0}^{(N_1+N_2)/2} \left\{ \left[J''_m(\lambda) + \frac{v}{\lambda} J'_m(\lambda) \right] - \frac{J_m(\lambda)}{I_m(\lambda)} \left[I''_m(\lambda) + \frac{v}{\lambda} I'_m(\lambda) \right] \right\} \times \left\{ A_m \cos\left(m\left[\alpha + \frac{(2\pi-\alpha)n_2}{N_2} \right] \right) + B_m \sin\left(m\left[\alpha + \frac{(2\pi-\alpha)n_2}{N_2} \right] \right) \right\} = 0 \quad \text{for } n_2 = 0, 1, \ldots, N_2. \tag{18}$$

Eq. (17) represents N_1 homogeneous algebraic equations in the unknowns $A_0, A_1, \ldots, A_{(N_1+N_2)/2}$ and $B_1, B_2, \ldots, B_{(N_1+N_2)/2}$, while Eq. (18) yields N_2+1 equations in those constants. The vanishing coefficient determinant represents the equation for the determination of the lower approximate eigenvalues λ. It yields the lower values λ_{mn} for a given magnitude of α in ascending order of which those compatible for a given mode m have to be selected between those of the clamped case and those of the totally simply supported case.

3.2. Clamped along part of the boundary and guided along the remainder

If the circular plate is clamped in the range $0<\varphi<\alpha$ and guided in the remaining boundary range $\alpha \leqslant \varphi \leqslant 2\pi$ (Fig. 1), the plate exhibits for the total range $0<\varphi \leqslant 2\pi$ a vanishing slope $\partial w/\partial r = 0$ at $r=a$. Before we proceed to this case we investigate the limit cases of a totally clamped plate and that of a totally guided plate. The totally clamped plate exhibits the eigenvalues as presented in Table 1 and Ref. [6]. For a plate totally guided at $r=a$ the eigenvalues are determined from the equation $\partial w/\partial r = 0$ and $V_r = 0$ at $r=a$ with the solution of the differential equation (1) as given by Eq. (12). The eigenvalues may then be determined from

$$2\lambda^3 J'_m(\lambda) - m^2(1-v)\left[J_m(\lambda) - \frac{J'_m(\lambda)}{I'_m(\lambda)} I_m(\lambda) \right] = 0 \tag{19}$$

and yield for $v=0.3$ the results given in Table 3, where the root zero ($m=0$, $n=1$) represents the translational rigid body motion. This is important to notice, since the counting of the roots has to be performed in an appropriate way in order to properly identify the mode for the mixed boundary case. For a partially clamped plate in the range $0<\varphi<\alpha$ and partially guided plate in the range $\alpha \leqslant \varphi \leqslant 2\pi$ we have to satisfy at $r=a$ for the deflection

$$W(r,\varphi) = \sum_{m=0}^{\infty} \left[J_m\left(\lambda \frac{r}{a}\right) - \frac{J'_m(\lambda)}{I'_m(\lambda)} I_m\left(\lambda \frac{r}{a}\right) \right] \{ A_m \cos m\varphi + B_m \sin m\varphi \}, \tag{20}$$

which satisfies the boundary condition $\partial w/\partial r = 0$ at $r=a$. The boundary condition $w=0$ for the clamped part of the plate, is

$$\sum_{m=0}^{\infty} \left[J_m(\lambda) - \frac{J'_m(\lambda)}{I'_m(\lambda)} I_m(\lambda) \right] (A_m \cos m\varphi + B_m \sin m\varphi) = 0 \tag{21}$$

Table 3
Eigenvalues $\lambda_{mn}^{(g)2}$ for a guided plate ($v=0.3$)

$n\backslash m$	0	1	2	3
1	0	3.0825	8.7849	16.9020
2	14.6820	28.3988	44.9041	64.1304
3	49.2185	72.8590	99.3610	128.6775
4	103.4995	137.0254	173.4422	212.7161

in the range $0<\varphi<\alpha$ and the boundary condition $V_r=0$ for the guided part

$$\sum_{m=0}^{\infty}\left\{2\lambda^3 J'_m(\lambda)-m^2(1-\nu)\left[J_m(\lambda)-\frac{J'_m(\lambda)}{I'_m(\lambda)}I_m(\lambda)\right]\right\}(A_m\cos m\varphi+B_m\sin m\varphi)=0$$
$$\text{in the range } \alpha\leqslant\varphi\leqslant 2\pi. \tag{22}$$

Following the same procedure as in Section 3.1 for the points at which these Eqs. (21) and (22) are to be satisfied yield the eigenvalues λ for a given Poisson ratio $\nu=0.3$.

3.3. Simply supported along part of the boundary and free along the remainder

For a circular plate with simply supported boundary in the range $0<\varphi<\alpha$ and free along the remaining boundary $\alpha\leqslant\varphi\leqslant 2\pi$, the common boundary condition of vanishing bending moment $M_r=0$ exists at $r=a$ for the total range $0\leqslant\varphi<2\pi$. The remaining partial boundary conditions $w=0$ at $r=a$ in the range $0<\varphi<\alpha$ and $V_r=0$ at $r=a$ in the range $\alpha\leqslant\varphi\leqslant 2\pi$ have now to be satisfied approximately by applying the above proposed method.

Before proceeding to this case of mixed boundary conditions we treat first the limiting cases where $\alpha=0$, i.e. the free plate, and $\alpha=2\pi$, i.e. the totally simply supported plate. The eigenvalues of the latter case were presented already in Table 2. For a completely free plate the eigenvalues are presented by

$$\{(1-\nu)\lambda J'_m(\lambda)+[\lambda^2-m^2(1-\nu)]J_m(\lambda)\}\{\lambda^3 I'_m(\lambda)-m^2(1-\nu)[\lambda I'_m(\lambda)-I_m(\lambda)]\}$$
$$+\{(1-\nu)\lambda I'_m(\lambda)-[\lambda^2+m^2(1-\nu)]I_m(\lambda)\}\{\lambda^3 J'_m(\lambda)+m^2(1-\nu)[\lambda J'_m(\lambda)-J_m(\lambda)]\}=0 \tag{23}$$

of which $\lambda_{mn}^{(f)2}$ is presented for $\nu=0.3$ in Table 4, where the roots zero ($m=0$, $n=1$ and $m=1$, $n=1$) represent the rigid body motions of translation and rotation, respectively. The deflection of the plate satisfying the boundary condition $M_r=0$ at $r=a$ for $0<\varphi\leqslant 2\pi$ yields with

$$\chi_m(\lambda;\nu)=\frac{(1-\nu)\lambda J'_m(\lambda)+[\lambda^2-m^2(1-\nu)]J_m(\lambda)}{(1-\nu)\lambda I'_m(\lambda)-[\lambda^2+m^2(1-\nu)]I_m(\lambda)}, \tag{24}$$

$$W(r,\varphi)=\sum_{m=0}^{\infty}\left[J_m\left(\lambda\frac{r}{a}\right)-\chi_m(\lambda;\nu)I_m\left(\lambda\frac{r}{a}\right)\right]\{A_m\cos m\varphi+B_m\sin m\varphi\}, \tag{25}$$

which when introduced in the remaining partial boundary condition results in

$$\sum_{m=0}^{\infty}[J_m(\lambda)-\chi_m(\lambda;\nu)I_m(\lambda)]\{A_m\cos m\varphi+B_m\sin m\varphi\}=0$$
$$\text{in the range } 0<\varphi<\alpha \tag{26}$$

Table 4
Eigenvalues $\lambda_{mn}^{(f)2}$ for a free plate ($\nu=0.3$)

$n\backslash m$	0	1	2	3
1	0	0	5.3583	12.4390
2	9.0031	20.4746	35.2601	53.0078
3	38.4432	59.8116	84.3662	111.9450
4	87.7502	118.9573	153.3059	190.6918

and

$$\sum_{m=0}^{\infty}\{\lambda^3 J_m'(\lambda)+m^2(1-\nu)[\lambda J_m'(\lambda)-J_m(\lambda)]+\chi_m(\lambda;\nu)[\lambda^3 I_m'(\lambda) - m^2(1-\nu)[\lambda I_m'(\lambda)-I_m(\lambda)]]\}\{A_m\cos m\varphi+B_m\sin m\varphi\}=0 \quad \text{in the range } \alpha\leqslant\varphi\leqslant 2\pi. \tag{27}$$

Following the above numerical procedure Eqs. (26) and (27) yield at arbitrary points on the boundary of the plate the approximate lower natural frequencies for the plate of simply supported-free mixed boundary conditions.

3.4. Free along part of the boundary and guided along the remainder

If part of the circular plate is free in the range $0<\varphi<\alpha$ and guided in the range $\alpha\leqslant\varphi\leqslant 2\pi$, the Kelvin–Kirchhoff edge reaction $V_r=0$ is valid along the total boundary $r=a$, while a mixed boundary condition exists and is described as $M_r=0$ in the range $0<\varphi<\alpha$ and $\partial w/\partial r=0$ in the range $\alpha\leqslant\varphi\leqslant 2\pi$. Before proceeding to this mixed boundary condition case, we first investigate the pure cases of a free boundary and that of a totally guided boundary of the circular plate. For the latter case the results are already presented in Table 3 for $\nu=0.3$. For a completely free boundary condition the eigenvalues are presented for $\nu=0.3$ in Table 4.

For the above described mixed boundary conditions the deflection satisfying the condition $V_r=0$ at $r=a$ and along the total range $0<\varphi\leqslant 2\pi$ yields with

$$\Phi_m(\lambda;\nu)=\frac{\lambda^3 J_m'(\lambda)+m^2(1-\nu)[\lambda J_m'(\lambda)-J_m(\lambda)]}{\lambda^3 I_m'(\lambda)-m^2(1-\nu)[\lambda I_m'(\lambda)-I_m(\lambda)]} \tag{28}$$

the expression

$$W(r,\varphi)=\sum_{m=0}^{\infty}\left[J_m\left(\lambda\frac{r}{a}\right)+\Phi_m(\lambda;\nu)I_m\left(\lambda\frac{r}{a}\right)\right]\{A_m\cos m\varphi+B_m\sin m\varphi\}, \tag{29}$$

which when introduced into the remaining mixed boundary conditions yields

$$\sum_{m=0}^{\infty}[J_m'(\lambda)+\Phi_m(\lambda;\nu)I_m'(\lambda)]\{A_m\cos m\varphi+B_m\sin m\varphi\}=0 \quad \text{in the range } \alpha\leqslant\varphi\leqslant 2\pi \tag{30}$$

and

$$\sum_{m=0}^{\infty}\{(1-\nu)\lambda J_m'(\lambda)+[\lambda^2-m^2(1-\nu)]J_m(\lambda)+\Phi_m(\lambda;\nu)[(1-\nu)\lambda I_m'(\lambda) - [\lambda^2+m^2(1-\nu)]I_m(\lambda)]\}\{A_m\cos m\varphi+B_m\sin m\varphi\}=0 \quad \text{in the range } 0<\varphi<\alpha. \tag{31}$$

With the above proposed numerical procedure we are able to determine the approximate lower natural frequencies of the circular plate with the above indicated mixed boundaries, if we choose a finite number of points on the boundary $r=a$.

3.5. Circular plate with partly clamped and free boundary

In the previous four cases the boundaries were always such that one of the two plate conditions had a common boundary condition along the total periphery at $r=a$. In many practical applications, however, partial boundary conditions may appear, which are all different in the assumed ranges, may it be by accidental failures or by design purposes. One case of particular interest is therefore the clamped plate for which part of the periphery has become loose, i.e. a free boundary, by structural failure. This case, however, is presenting a more involved numerical evaluation procedure, as shall be performed in the following treatment.

Before treating this mixed boundary case we shall first recall the eigenvalues for a completely clamped plate and for a completely free plate. In these cases, the eigenvalues $\lambda_{mn}^{(f)2}$ are presented for a totally free plate in Table 4, while for a completely clamped plate the eigenvalues $\lambda_{mn}^{(c)2}$ are given in Table 1 and Ref. [6]. It should be noted that Ref. [6] does not present roots for all boundary value cases presented here.

The solution of the plate satisfying the plate Eq. (1) is given by

$$W(r,\varphi)=\sum_{m=0}^{\infty}\left\{\left[A_mJ_m\left(\lambda\frac{r}{a}\right)+B_mI_m\left(\lambda\frac{r}{a}\right)\right]\cos m\varphi\left[C_mJ_m\left(\lambda\frac{r}{a}\right)+D_mI_m\left(\lambda\frac{r}{a}\right)\right]\sin m\varphi\right\}, \tag{32}$$

which yields with the boundary conditions in the two ranges the equations

$$\sum_{m=0}^{\infty}\{[A_mJ_m(\lambda)+B_mI_m(\lambda)]\cos m\varphi+[C_mJ_m(\lambda)+D_mI_m(\lambda)]\sin m\varphi\}=0 \quad \text{in the range } 0<\varphi<\alpha, \tag{33}$$

$$\sum_{m=0}^{\infty}\{[A_m\lambda J_m'(\lambda)+B_m\lambda I_m'(\lambda)]\cos m\varphi+[C_m\lambda J_m'(\lambda)+D_m\lambda I_m'(\lambda)]\sin m\varphi\}=0 \quad \text{in the range } 0<\varphi<\alpha \tag{34}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty}\{&[A_m(\lambda^2J_m''(\lambda)+v\lambda J_m'(\lambda)-m^2vJ_m(\lambda))+B_m(\lambda^2I_m''(\lambda)+v\lambda I_m'(\lambda)\\&-m^2vI_m(\lambda))]\cos m\varphi+[C_m(\lambda^2J_m''(\lambda)+v\lambda J_m'(\lambda)-m^2vJ_m(\lambda))+D_m(\lambda^2I_m''(\lambda)\\&+v\lambda I_m'(\lambda)-m^2vI_m(\lambda))]\sin m\varphi\}=0 \quad \text{in the range } \alpha\leqslant\varphi\leqslant 2\pi,\end{aligned} \tag{35}$$

$$\begin{aligned}\sum_{m=0}^{\infty}\{&[A_m(\lambda^3J_m'''(\lambda)+\lambda^2J_m''(\lambda)-[m^2(2-v)+1]\lambda J_m'(\lambda)+m^2(3-v)J_m(\lambda))\\&+B_m(\lambda^3I_m'''(\lambda)+\lambda^2I_m''(\lambda)-[m^2(2-v)+1]\lambda I_m'(\lambda)+m^2(3-v)I_m(\lambda))]\cos m\varphi\\&+[C_m(\lambda^3J_m'''(\lambda)+\lambda^2J_m''(\lambda)-[m^2(2-v)+1]\lambda J_m'(\lambda)+m^2(3-v)J_m(\lambda))\\&+D_m(\lambda^3I_m'''(\lambda)+\lambda^2I_m''(\lambda)-[m^2(2-v)+1]\lambda I_m'(\lambda)+m^2(3-v)I_m(\lambda))]\sin m\varphi\}=0\\&\text{in the range } \alpha\leqslant\varphi\leqslant 2\pi.\end{aligned} \tag{36}$$

Satisfying these four equations at arbitrary φ and truncating the infinite series such that the algebraic system exhibits as many equations as coefficients A_m, B_m, C_m and D_m yields a system of algebraic equations; setting the coefficient determinant equal to zero leads to an approximate transcendental eigenvalue equation. The roots of this determinant represent the approximate natural frequencies of the above circular plate with the given mixed boundary conditions. With $\varphi=\alpha n_1/(N_1+1)$, $n_1=1,2,\ldots,N_1$, $\varphi=\alpha+(2\pi-\alpha)n_2/N_2$, $n_2=0,1,2,\ldots,N_2$ Eqs. (33) and (34) yield each N_1 algebraic equations, while Eqs. (35) and (36) result each in (N_2+1) algebraic equations. The infinite series has to be truncated with $m=0$ to $m=(N_1+N_2)/2$, where (N_1+N_2) must be even, to represent an algebraic system of $2(N_1+N_2+1)$ equations for the unknown constants A_0, A_1, $A_2,\ldots,A_{(N_1+N_2)/2}$, B_0, B_1, $B_2,\ldots,B_{(N_1+N_2)/2}$, C_0, C_1, $C_2,\ldots,C_{(N_1+N_2)/2}$ and D_0, D_1, $D_2,\ldots,D_{(N_1+N_2)/2}$.

It should be also mentioned that a circular plate with more than two mixed boundary conditions at its periphery $r=a$ may also be treated, if one is willing to solve the appearing larger order determinants numerically.

3.6. Circular plate with partly simply supported and guided boundary

Before treating this mixed boundary condition case we shall recall the eigenvalues of the completely simply supported plate (Table 2) and those of the totally guided plate (Table 3). These indicate the values of λ^2 on the ordinate for $\alpha = 0$ and $\alpha = 2\pi$. We detect that for $\alpha = 2\pi$, i.e. a completely guided plate, the axisymmetric mode $m = 0$, $n = 1$ starts from the abscissa axis, representing a translational rigid body motion.

The solution of the plate satisfying Eq. (1) is presented by Eq. (32), which yields with the boundary conditions in the two ranges, Eqs. (33) and (35) in the range $0<\varphi<\alpha$, and Eqs. (34) and (36) for the range $\alpha \leqslant \varphi \leqslant 2\pi$ the equations for the determination of λ^2. Satisfying these four equations at freely chosen point φ and truncating the infinite series yields a system for the determination of the lower approximate eigenvalues, i.e. the vanishing coefficient determinant as approximate eigenvalue equation as has been shown above.

3.7. Other boundary value cases

The procedure may well be applied to cases, where the boundary of the plate exhibits more than two mixed boundary conditions. If the boundary of the plate shows three different boundary conditions, say clamped in the range $0 \leqslant \varphi \leqslant \alpha$, simply supported in the range $\alpha < \varphi \leqslant \beta$, $(\alpha < \beta < 2\pi)$, and free in the remaining angular range $\beta < \varphi < 2\pi$, then we have to satisfy in the first range the boundary conditions (9a), in the second range (9b) and in the third range those given by Eqs. (9c). The involved numerical procedure, however, may be—in spite of coinciding equal boundary conditions—quite cumbersome and requires the solution of a large number of equations, i.e. a high order determinant for the determination of the natural frequencies.

The solution of the plate satisfying the plate equation (1) is given by Eq. (32). Satisfying in the range $0 \leqslant \varphi < \alpha$ yields for $w = 0$ and $\partial w/\partial r = 0$ the equations

$$\sum_{m=0}^{\infty}\left\{[A_m J_m(\lambda) + B_m I_m(\lambda)]\cos\left(m\frac{j\alpha}{N_1}\right) + [C_m J_m(\lambda) + D_m I_m(\lambda)]\sin\left(m\frac{j\alpha}{N_1}\right)\right\} = 0, \tag{37}$$

where $j = 0, 1, \ldots, (N_1 - 1)$ and $0 \leqslant \varphi_j < \alpha_j/N_1$. These are N_1 algebraic and homogeneous equations. In addition we obtain

$$\sum_{m=0}^{\infty}\left\{[A_m J'_m(\lambda) + B_m I'_m(\lambda)]\lambda\cos\left(m\frac{j\alpha}{N_1}\right) + [C_m J'_m(\lambda) + D_m I'_m(\lambda)]\lambda\sin\left(m\frac{j\alpha}{N_1}\right)\right\} = 0 \tag{38}$$

i.e. N_1 algebraic equations.

For the range $\alpha \leqslant \varphi < \beta$ the plate exhibits the simply supported boundary conditions $w = 0$ and $M_r = 0$ at $r = a$. This yields with $k = 0, 1, 2, \ldots, (N_2 - 1)$ and $\alpha < \varphi_k < \beta$ two additional systems of algebraic equations, totalling $2N_2$ equations. They are

$$\begin{aligned}\sum_{m=0}^{\infty}\Bigg\{&[A_m J_m(\lambda) + B_m I_m(\lambda)]\cos\left[m\left(\alpha + \frac{(\beta-\alpha)k}{N_2}\right)\right]\\ &+ [C_m J_m(\lambda) + D_m I_m(\lambda)]\sin\left[m\left(\alpha + \frac{(\beta-\alpha)k}{N_2}\right)\right]\Bigg\} = 0,\end{aligned} \tag{39}$$

and

$$\begin{aligned}\sum_{m=0}^{\infty}\Big\{&[A_m(\lambda^2 J''_m(\lambda) + v\lambda J'_m(\lambda) - m^2 v J_m(\lambda))\\ &+ B_m(\lambda^2 I''_m(\lambda) + v\lambda I'_m(\lambda) - m^2 v I_m(\lambda))]\cos\left[m\left(\alpha + \frac{(\beta-\alpha)k}{N_2}\right)\right]\\ &+ [C_m(\lambda^2 J''_m(\lambda) + v\lambda J'_m(\lambda) - m^2 v J_m(\lambda))\\ &+ D_m(\lambda^2 I''_m(\lambda) + v\lambda I'_m(\lambda) - m^2 v I_m(\lambda))]\sin\left[m\left(\alpha + \frac{(\beta-\alpha)k}{N_2}\right)\right]\Big\} = 0.\end{aligned} \tag{40}$$

For the remaining boundary condition range $\beta \leqslant \varphi_l < 2\pi$ the plate behaves like a guided plate, which is described by the boundary condition $\partial w/\partial r = 0$ and $V_r = 0$ at $r = a$. This results with $l = 0, 1, 2, \ldots, (N_3 - 1)$ and $\varphi_l = \beta + ((2\pi - \beta)l)/N_3$ in $2N_3$ algebraic equations. They are given by

$$\sum_{m=0}^{\infty} \left\{ [A_m J'_m(\lambda) + B_m I'_m(\lambda)]\lambda \cos\left[m\left(\alpha + \frac{(\beta-\alpha)l}{N_3}\right)\right] + [C_m J'_m(\lambda) + D_m I'_m(\lambda)]\lambda \sin\left[m\left(\alpha + \frac{(\beta-\alpha)l}{N_3}\right)\right] \right\} = 0, \quad (41)$$

and

$$\sum_{m=0}^{\infty} \left\{ [A_m \Omega_m + B_m \Psi_m] \cos\left[m\left(\alpha + \frac{(\beta-\alpha)l}{N_3}\right)\right] + [C_m \Omega_m + D_m \Psi_m] \sin\left[m\left(\alpha + \frac{(\beta-\alpha)l}{N_3}\right)\right] \right\} = 0, \quad (42)$$

where Ω_m represents the expression in the first round parenthesis, Ψ_m that in the second round parenthesis of Eq. (36).

If the mixed boundary conditions contain an elastically supported boundary, then we have to apply the conditions given in Eqs. (9e) for that particular range. For an elastic plate with a clamped boundary in the range $0 \leqslant \varphi \leqslant \alpha$ and an elastically supported boundary in the remaining range $\alpha < \varphi < 2\pi$, we have to satisfy the boundary condition $w = \partial w/\partial r = 0$ in the range $0 \leqslant \varphi \leqslant \alpha$ and $M_r - K\,\partial w/\partial r = 0$ and $V_r + kw = 0$ at $r = a$ and in the range $\alpha < \varphi < 2\pi$. This would in comparison with Section 3.1 require the solution of the slightly different algebraic system of equations, which contains in Eqs. (35) and (36) additional terms $-K\,\partial w/\partial r$ and kw, respectively.

Other cases of two mixed interchanging boundary conditions may also be treated. As an example we consider a mixed boundary condition case being (Fig. 1)

$$\text{clamped: } w = 0 \quad \text{and} \quad \frac{\partial w}{\partial r} = 0 \ \text{ at } r = a \ \text{ in the ranges} \begin{Bmatrix} 0 \leqslant \varphi \leqslant \frac{\pi}{2} \\ \pi \leqslant \varphi \leqslant \frac{3\pi}{2} \end{Bmatrix} \quad (43)$$

and

$$\text{simply supported: } w = 0 \quad \text{and} \quad M_r = 0 \ \text{ at } r = a \ \text{ in the ranges} \begin{Bmatrix} \frac{\pi}{2} < \varphi < \pi \\ \frac{3\pi}{2} < \varphi < 2\pi \end{Bmatrix}. \quad (44)$$

Over the complete boundary we observe $w = 0$ at $r = a$ which renders the solution (12), while the boundary condition $\partial w/\partial r = 0$ at $r = a$ are in the ranges given by Eq. (43) yields with $\varphi = \pi n_1/2(N_1 + 1)$, $n_1 = 1, 2, \ldots, N_1$ and $\varphi = \pi + \pi n_2/2N_2$, $n_2 = 1, 2, \ldots, N_2$ the expressions

$$\sum_{m=0}^{(N_1+N_2+N_3+N_4)/2} \left[J'_m(\lambda) - \frac{J_m(\lambda)}{I_m(\lambda)} I'_m(\lambda)\right] \left\{ A_m \cos\left(\frac{m\pi n_1}{2N_1}\right) + B_m \sin\left(\frac{m\pi n_1}{2N_1}\right) \right\} = 0 \quad \text{for } n_1 = 1, 2, \ldots, N_1 \quad (45)$$

representing N_1 equations, and

$$\sum_{m=0}^{(N_1+N_2+N_3+N_4)/2} \left[J'_m(\lambda) - \frac{J_m(\lambda)}{I_m(\lambda)} I'_m(\lambda)\right] \left\{ A_m \cos m\left(\pi + \frac{\pi n_2}{2N_2}\right) + B_m \sin m\left(\frac{\pi n_2}{2N_2}\right) \right\} = 0 \quad \text{for } n_2 = 1, 2, \ldots, N_2 \quad (46)$$

representing N_2 equations, respectively. The simply supported parts of the boundary (see Eq. (44)), as obtained from Eq. (16) are then given by ($\varphi = \pi/2 + \pi n_3/2N_3$, $n_3 = 0, 1, 2, \ldots, N_3$, $\varphi = 3\pi/2 + \pi n_4/2N_4$,

$n_4 = 1, 2, \ldots, N_4$)

$$\sum_{m=0}^{(N_1+N_2)/2} \left\{ \left[J''_m(\lambda) + \frac{v}{\lambda} J'_m(\lambda) \right] - \frac{J_m(\lambda)}{I_m(\lambda)} \left[I''_m(\lambda) + \frac{v}{\lambda} I'_m(\lambda) \right] \right\} \times \left\{ A_m \cos\left(m \left[\frac{\pi}{2} + \frac{\pi n_3}{2N_3} \right] \right) + B_m \sin\left(m \left[\frac{\pi}{2} + \frac{\pi n_3}{2N_3} \right] \right) \right\} = 0 \quad \text{for } n_3 = 0, 1, \ldots, N_3 \tag{47}$$

representing $N_3 + 1$ equations, and

$$\sum_{m=0}^{(N_1+N_2)/2} \left\{ \left[J''_m(\lambda) + \frac{v}{\lambda} J'_m(\lambda) \right] - \frac{J_m(\lambda)}{I_m(\lambda)} \left[I''_m(\lambda) + \frac{v}{\lambda} I'_m(\lambda) \right] \right\} \times \left\{ A_m \cos\left(m \left[\frac{3\pi}{2} + \frac{\pi n_4}{2N_4} \right] \right) + B_m \sin\left(m \left[\frac{3\pi}{2} + \frac{\pi n_4}{2N_4} \right] \right) \right\} = 0 \quad \text{for } n_4 = 1, \ldots, N_4 \tag{48}$$

representing N_4 equations. The above Eqs. (45)–(48) represent a homogeneous algebraic system of $(N_1 + N_2 + N_3 + N_4 + 1)$ equations in the unknowns $A_0, A_1, \ldots, A_{(N_1+N_2+N_3+N_4)/2}, B_1, B_2, \ldots, B_{(N_1+N_2+N_3+N_4)/2}$. We just have to observe that $(N_1 + N_2 + N_3 + N_4)$ is an even number.

4. Numerical evaluations and conclusions

Some of the above obtained results have been evaluated numerically for the lower modes of a circular plate. The natural frequencies $\lambda_{mn}^2 = \omega_{mn} a^2 \sqrt{\varrho h/D}$ are presented for the mixed boundary condition of a "simply supported–clamped" plate as a function of the angle α/π and $v = 0.3$ in Fig. 2. They may also be found together with the nodal lines for the asymmetric case ($m \neq 0$) in Ref. [5]. For $\alpha = 0$ we deal with a plate of a purely simply supported boundary condition, while for $\alpha = 2\pi$ the boundary is in a purely clamped state. These values are indicated for $v = 0.3$ in the figure as $\circ$ for the simply supported boundary and as $\otimes$ for the clamped boundary. The influence of the Poisson ratio v, which appears only in the simply supported boundary is indicated by the $*$-star sign, where the upper star represents the λ^2-values for $v = 0.5$ and the lower one that of $v = 0.2$. The natural frequency for $m = 0$, $n = 1$ increases from 4.9351 to 10.2158 and exhibits a slight curvature close to $\alpha = \pi$, i.e. the case where one half of the plate is simply supported and the other half is clamped. The numerical results were compared with those of Refs. [3,4], where only the mode $m = 0$, $n = 1$ has been treated and presented. The values presented above show close results to those given in Ref. [3]. In the numerical evaluation of our treatment we employed $N_1 + N_2 = 150$, and vary N_1 and N_2 according to the magnitude of α, which means a varying ratio of $N_2/N_1 = \bar{r}$, which decreases as the number of points N_2 considered simply supported decreases and the number of points N_1 considered clamped increases. The magnitudes of N_1 and N_2 are chosen such that an increase does not affect the accuracy of the plotted λ^2-values (ω). If α is small the portion of the boundary being clamped is small and needs only a small number N_1, while that of the large boundary region being simply supported requires a large number N_2 (adding up to $N_1 + N_2 = 150$) for the numerical procedure. As the clamped portion increases so does N_1, while N_2 decreases. Our treatment of the problem includes also the not yet treated axisymmetric second mode $m = 0$, $n = 2$, starting for a totally simply supported plate from $(\lambda_{02}^{(ss)})^2 = 29.72$ and reaches for a completely clamped plate the value $(\lambda_{02}^{(c)})^2 = 39.77$ (see also Tables 1 and 2). We notice increased varying curvature for $\alpha \neq 0$ and $\alpha \neq 2\pi$. The results for asymmetric modes $m \neq 0$ exhibit for the first modal number, i.e. $m = n = 1$ two values λ_{11}^2 and $\lambda_{11}'^2$ for $\alpha \neq 0$ and $\alpha \neq 2\pi$. In the pure boundary condition case we observe the magnitudes $(\lambda_{11}^{(ss)})^2 = 13.8982$ and $(\lambda_{11}^{(c)})^2 = 21.26$. Such asymmetric cases have not yet been treated previously. In the mixed boundary condition case the modal part ① exhibits in the range $0 < \alpha < \pi$ eigenfrequencies ω_{11} being smaller than those of the branch ②, while for $\alpha > \pi$, i.e. the case, where more than half of the plate is clamped at its boundary, the eigenfrequency ω_{11} of the modal part ① is larger than that of the branch ②. These two branches caused by the mixing of boundaries shall also exhibit different nodal lines (see Ref. [5] for details).

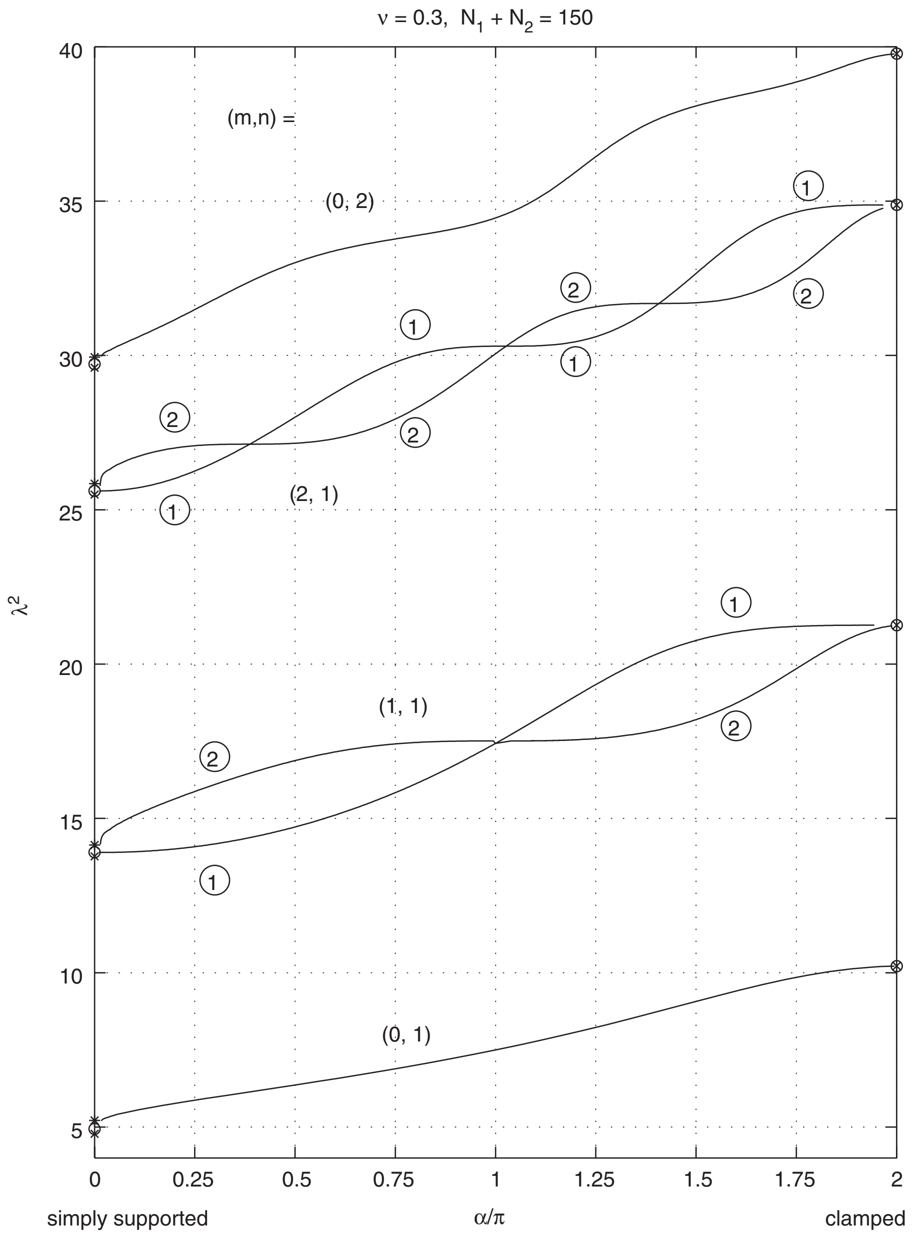

Fig. 2. Modal frequencies of clamped–simply supported plate.

The mode $m = 2$, $n = 1$ which starts for a completely simply supported plate at $(\lambda_{21}^{(ss)})^2 = 25.61$ and reaches for a totally clamped plate the magnitude $(\lambda_{21}^{(c)})^2 = 34.877$, exhibits also two branches ① and ② for $\alpha \neq 0, 2\pi$ with increased fluctuations and flexions (see Fig. 2).

The following mixed boundary condition cases have not been treated previously and reveal interesting vibrational behavior as a function of the magnitude of the angle α. For a plate for which part of the boundary condition at $r = a$ is described as clamped and the remainder of it is guided, some results are presented in Fig. 3. We notice that the axisymmetric mode $m = 0$, $n = 1$ exhibits for the square of the eigenvalue a range from $(\lambda_{01}^{(g)})^2 \approx 1.45$ to $(\lambda_{01}^{(c)})^2 = 10.2158$. It should be noted that the square of the eigenvalue for the mode $m = 0$, $n = 1$ does not approach the square of the eigenvalue $(\lambda_{01}^{(g)})^2 = 0$ of the totally guided plate when $\alpha \to 0$, because the limit case for $\alpha \to 0$ is not a totally guided plate, but a plate guided all over the boundary

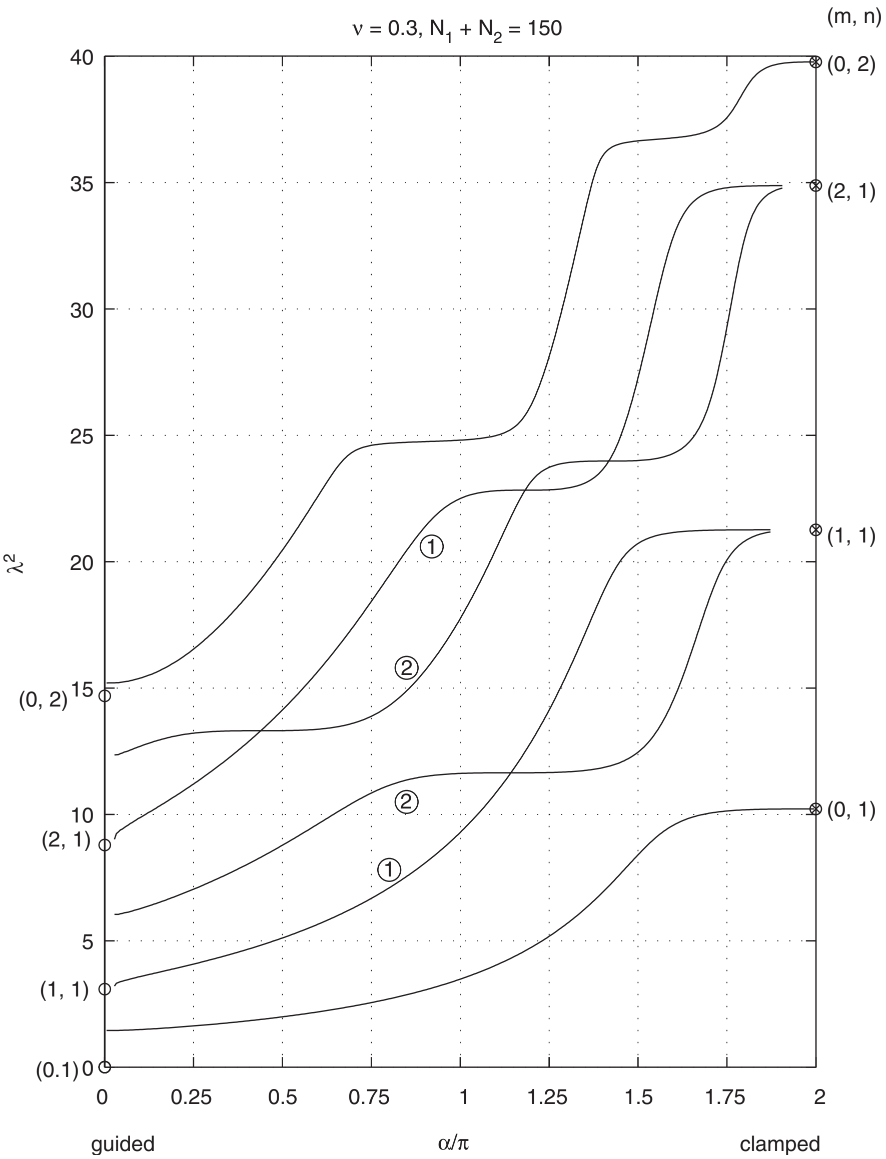

Fig. 3. Modal frequencies of clamped–guided plate.

with exception at point $(r,\varphi)=(a,0)$, where the boundary of the plate is fixed. The course of the magnitude of the mixed case results in natural frequencies between $\omega_{01}=1.45/a^2\sqrt{\varrho h/D}$ and $\omega_{01}=10.2158/a^2\sqrt{\varrho h/D}$. For the next axisymmetric mode $m=0$, $n=2$ the magnitudes λ^2 fluctuate between $(\lambda_{02}^{(g)})^2\approx 15.2$ and $(\lambda_{02}^{(c)})^2=39.77$ and exhibits increased flexion. Again for $\alpha\to 0$ the square of the eigenvalue does not approach that of the totally guided plate, but to a value that is about 3.5% higher. The asymmetric mode $m=n=1$ again shows for the mixed boundary conditions two branches ① and ② which intersect and show some flexion with increasing α. Both branches approach for $\alpha\to 2\pi$ to the limit case of totally clamped plate, i.e. the square of the eigenvalue $(\lambda_{11}^{(c)})^2=21.26$. When $\alpha\to 0$ only one branch approaches to $(\lambda_{11}^{(g)})^2=3.08$. This branch corresponds obviously to the $\sin\varphi$-solution, while the other one corresponding to the $\cos\varphi$-solution has a limit value of about 6.0 for the square of the eigenvalue, which is twice as large as the first one. Similar to this are the results for the mode $m=2$, $n=1$. The eigenvalues for the mixed boundary condition case "simply supported–free" are exhibited in Fig. 4 for $(m,n)=(0,1)$, (0,2), (1,1) and (2,1). Since the plate with a totally

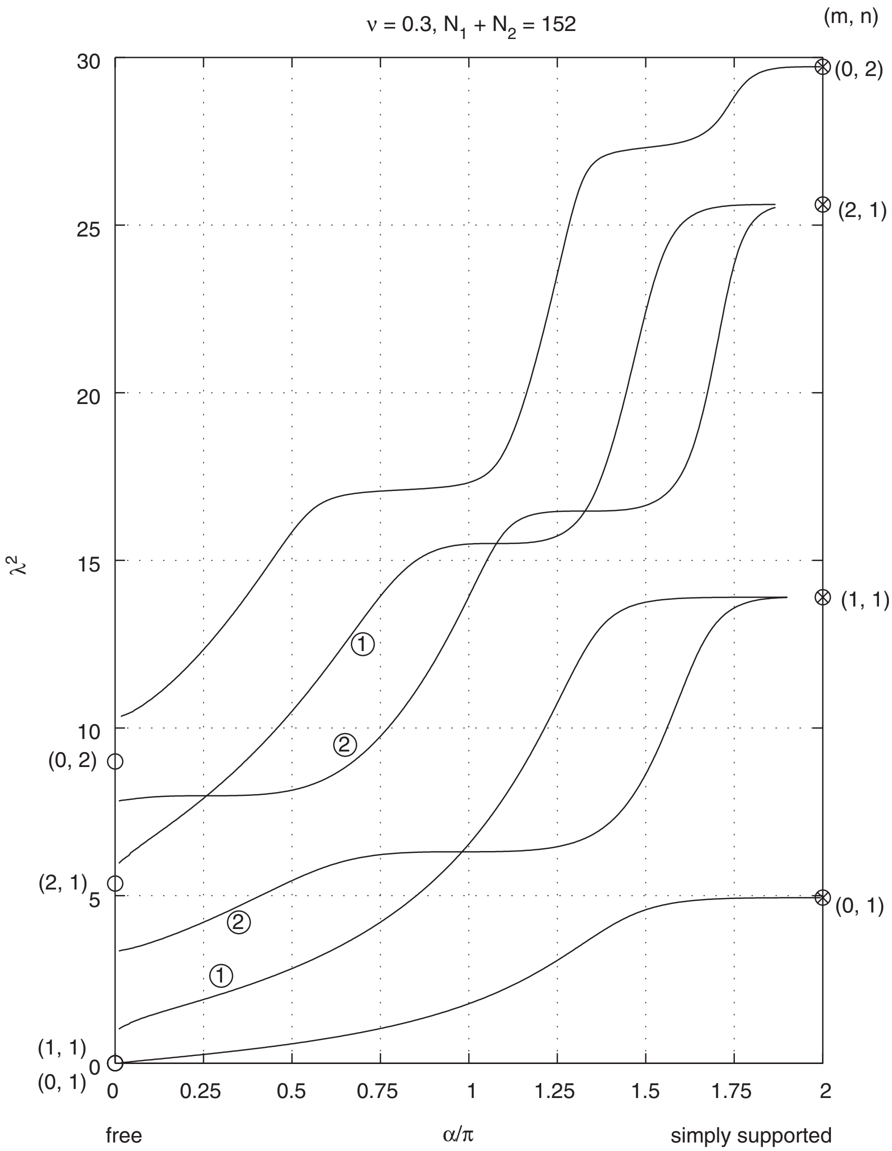

Fig. 4. Modal frequencies of simply supported–free plate.

free boundary is capable to perform rigid body motion in translation as well as in rotation, the curve belonging to the mode $m = 0$, $n = 1$ starts from $\lambda^2 = 0$. Otherwise, the results show similar effects as in the previous cases, i.e. increased flexion and curvature with increased mode number m and for asymmetric modes two frequency ranges ① and ②.

For the mixed boundary condition case free–guided the eigenvalues are presented in Fig. 5. The axisymmetric mode $m = 0$, $n = 1$ is presented by the abscissa axis, exhibiting the rigid body motion in translation, indicating that the axisymmetric mode $m = 0$, $n = 1$ is nothing but an up and down translatory motion of the rigid plate. It is independent of the magnitude of α, while the mode $m = n = 1$ shows for $\alpha = 2\pi$, i.e. a completely free boundary for rotational rigid body motion. This means that a rotational rigid body motion of the totally free plate is representing at $\alpha = 2\pi$ the asymmetric mode $m = 1$, $n = 1$. For $\alpha \neq 2\pi$ we detect again two branches of eigenvalues. Again we obtain for axisymmetric modes with $m = 0$ one eigenvalue for all α-values, as may be seen for $m = 0$ and $n = 1$, 2 in Fig. 5. For the mode $m = 2$, $n = 1$ the eigenvalues exhibit again two branches of different flexions, curvatures and fluctuations as α increases.

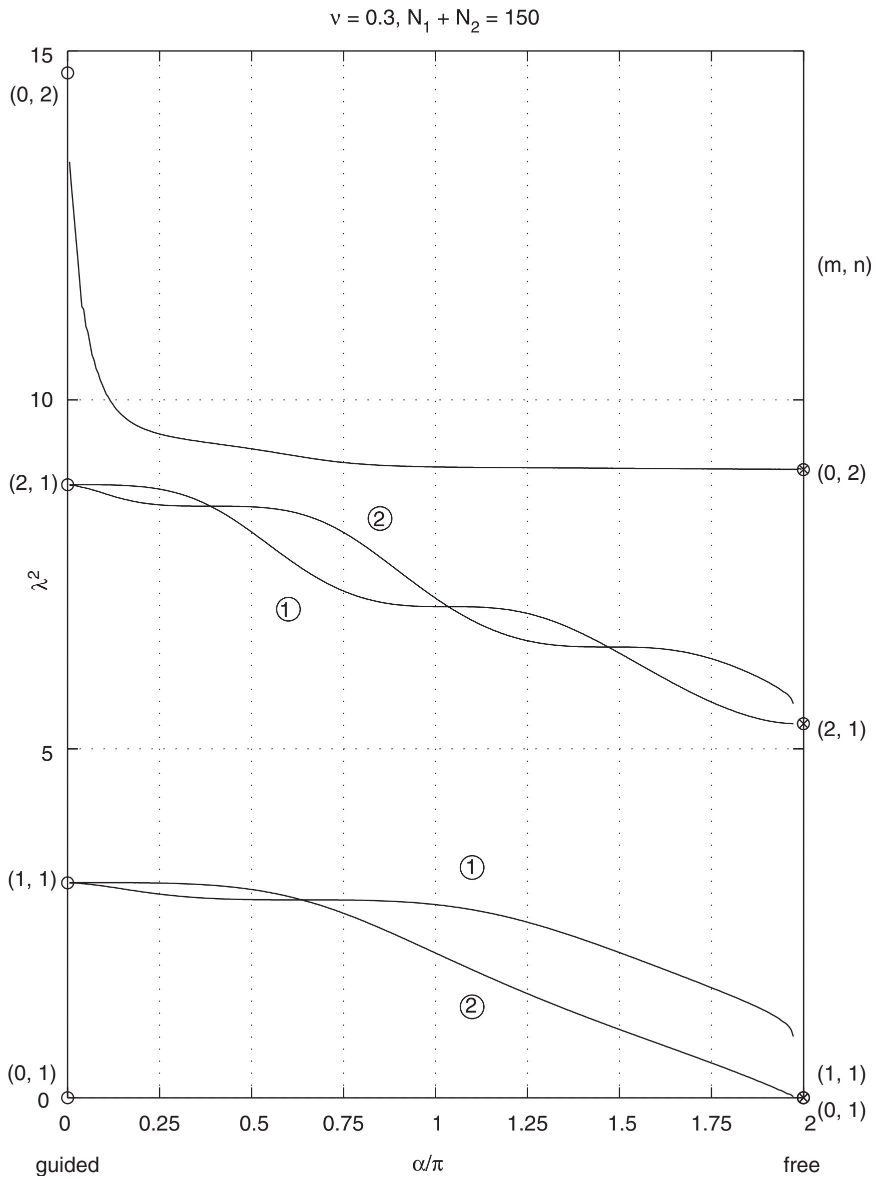

Fig. 5. Modal frequencies of free–guided plate.

The mixed boundary conditions clamped–free requires for the determination of the approximate squares of the eigenvalues λ^2 the solution of a set of algebraic equations stemming from Eqs. (33)–(36), where the set of Eqs. (33) and (34) have to be satisfied in the clamped boundary range $0<\varphi<\alpha$, while the set of Eqs. (35) and (36) satisfy a finite number of free boundary values in the range $\alpha\leqslant\varphi\leqslant 2\pi$. The results are given for $(m,n)=(0,1)$, $(m,n)=(0,2)$ and $m=n=1$ in Fig. 6 for $\nu=0.3$. The axisymmetric mode $m=0$, $n=1$ exhibits as the asymmetric mode $m=n=1$ for $\alpha=0$, i.e. a totally free plate, the magnitude $\lambda^2=0$ as mentioned above. As α increases to $\alpha=2\pi$ the axisymmetric mode exhibits increasing λ^2 and reaches finally at $\alpha=2\pi$ a magnitude of $\lambda^2=10.2158$. It may be noticed that the magnitude of λ^2 increases more rapidly above a region in which more than half of the plate exhibits a clamped boundary. For the asymmetric mode $m=n=1$ the two branches of λ^2 cross each other again shortly above $\alpha=\pi$ and exhibit similar behavior as in the above cases. For the axisymmetric mode $m=0$, $n=2$ the course of λ^2 may be seen in Fig. 6, originating

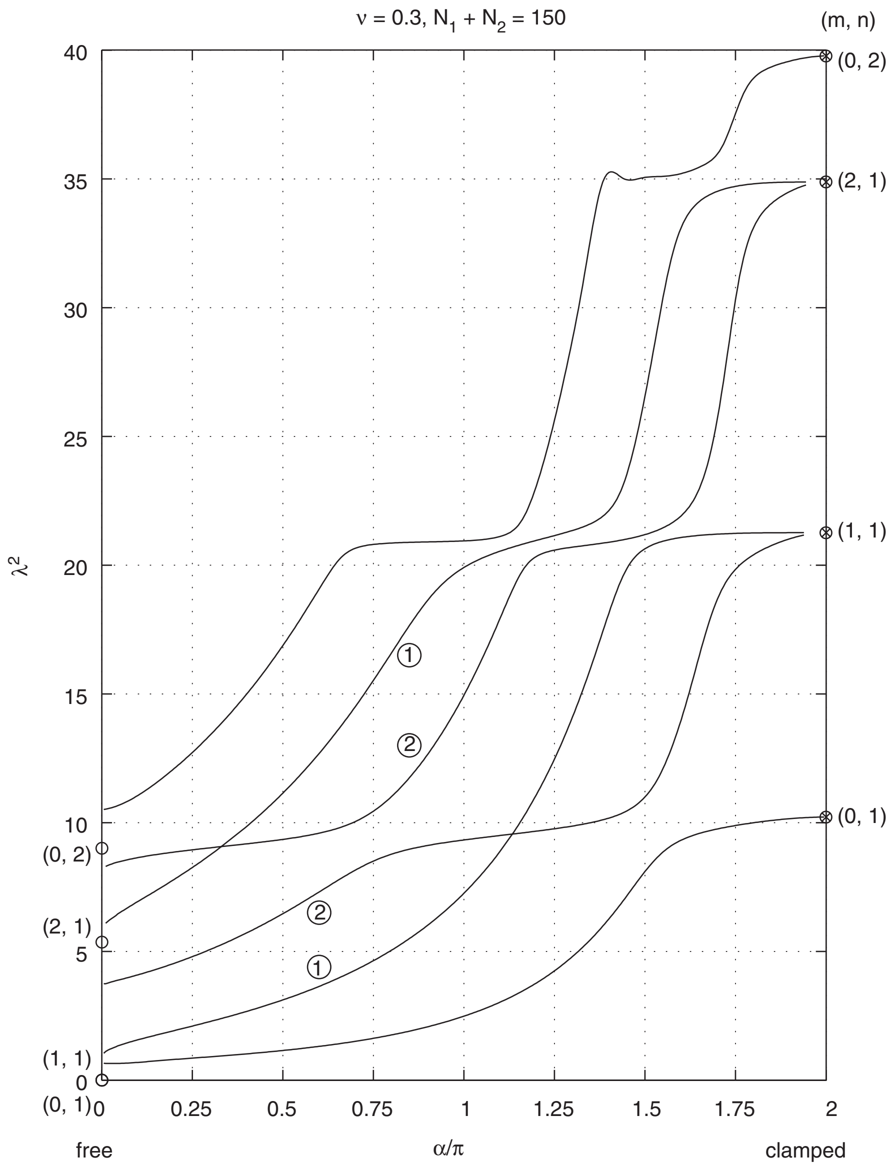

Fig. 6. Modal frequencies of clamped–free plate.

for $\alpha = 0$ at $\lambda^2 \approx 10.5$ and reaches with varying curvature the value of a completely clamped plate ($\alpha = 2\pi$), i.e. the magnitude $\lambda^2 = 39.77$.

The results λ^2 for the mixed boundary conditions simply supported–guided are presented in Fig. 7. The values for a completely simply supported plate and a completely guided plate are presented as ⊗ and ○ marks at $\alpha = 2\pi$ or $\alpha = 0$, respectively (see also Tables 2 and 3). Again we notice that the axisymmetric modes $m = 0$ ($n = 1, 2$) are represented by one frequency each, which exhibits with increasing n stronger variations as α increases. The asymmetric oscillation frequencies $m \neq 0$, i.e. $m = 1$ and $m = 2$, show again two branches ① and ②, as indicated in Fig. 7. If the plate is totally guided the mode $m = 0$, $n = 1$ is only capable to perform a rigid body translation, as indicated by the o-mark at $\lambda^2 = 0$, in spite of the fact that our analysis, which is only valid till shortly before $\alpha \to 0$ represents a finite value (same reason as explained above). The same is true for the other modes $m \neq 0$. It may be noticed in contrast to all other cases above that for $\pi < \alpha < 2\pi$ the value of λ^2 assumes values larger than those of the pure simply supported plate ⊗.

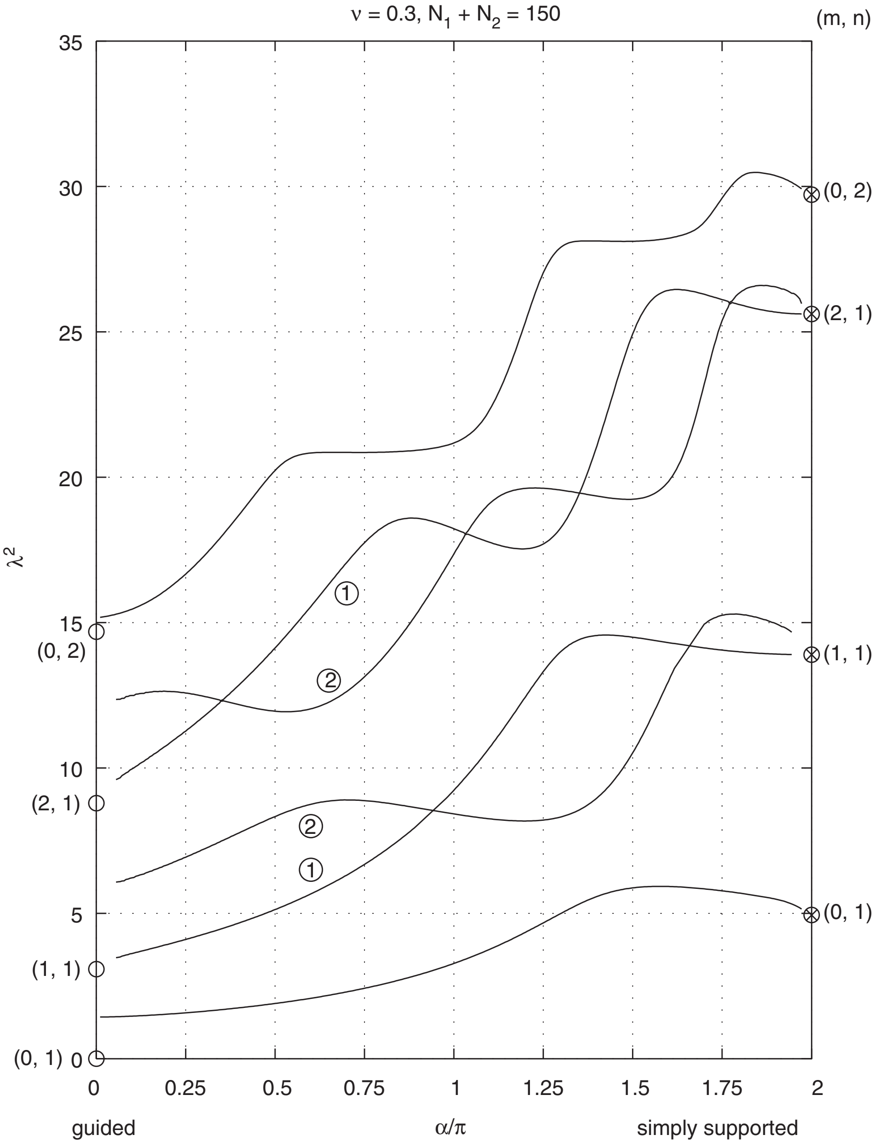

Fig. 7. Modal frequencies of simply supported–guided plate.

The proposed and executed method of satisfying the mixed boundary regions at appropriately chosen points on the boundary $r = a$ of the plate—at various angular locations—i.e. points on $r = a$ at $\varphi = \alpha$, raises of course the question of convergence and accuracy of the approximate natural frequencies of the here treated lower vibration modes for mixed boundary condition cases. How close or distant should the points at which the boundary conditions are satisfied be chosen in order to obtain an acceptable engineering value for the natural frequencies? To answer this question and obtain some engineering confidence in the above method we have determined the square of the eigenvalues λ, i.e. $\lambda^2 \sim \omega$, for various arrangements of the chosen points N_1 and N_2 where N_1 and N_2 have been chosen in proportionality to the magnitude of the α-values. Table 5a exhibits for $\alpha = \pi/2$ and for the clamped–simply supported mixed boundary case the results for λ^2 with varying $(N_1 + N_2)$ magnitude from $N_1 + N_2 = 50$ to 340 in steps of 10. The value $N_1 + N_2 = 150$, used in our numerical evaluations throughout has been bold typed. It may be noticed that the increase from $N_1 + N_2 =$ 150 to $N_1 + N_2 = 340$ resulted only in a change of λ^2 of about $\frac{1}{4}\%$, which is less than the thickness of the curve $m = 0$, $n = 1$ in Fig. 2. Even in the case of $N_1 + N_2 = 150$ the error appearing in the axisymmetric natural

Table 5a
Eigenvalues λ^2 for various modes (m, n): boundary conditions clamped–simply supported

N_1+N_2	N_1	N_2	$\alpha=\pi/2$						π	$3\pi/2$
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②	(0,1)	(0,1)
50	37	13	6.314	32.903	14.604	16.765	27.723	27.142	7.430	8.984
60	44	16	6.323	32.923	14.630	16.789	27.784	27.144	7.445	9.002
70	52	18	6.332	32.944	14.654	16.811	27.839	27.146	7.456	9.017
80	59	21	6.338	32.956	14.668	16.824	27.872	27.147	7.465	9.028
90	67	23	6.344	32.968	14.682	16.837	27.905	27.149	7.472	9.038
100	74	26	6.347	32.976	14.691	16.845	27.925	27.150	7.478	9.045
110	82	28	6.351	32.985	14.700	16.853	27.947	27.151	7.483	9.051
120	89	31	6.354	32.990	14.706	16.859	27.961	27.152	7.487	9.056
130	97	33	6.357	32.996	14.713	16.865	27.977	27.153	7.490	9.061
140	104	36	6.358	33.000	14.718	16.869	27.987	27.154	7.493	9.064
150	112	38	6.361	33.005	14.723	16.873	27.998	27.155	7.496	9.068
160	119	41	6.362	33.008	14.726	16.876	28.006	27.156	7.498	9.071
170	127	43	6.364	33.012	14.730	16.880	28.015	27.156	7.500	9.074
180	134	46	6.365	33.014	14.733	16.882	28.021	27.157	7.502	9.076
190	142	48	6.367	33.017	14.736	16.885	28.028	27.157	7.504	9.078
200	149	51	6.368	33.019	14.738	16.887	28.034	27.158	7.505	9.080
210	157	53	6.369	33.022	14.741	16.889	28.039	27.158	7.507	9.082
220	164	56	6.370	33.024	14.743	16.890	28.043	27.159	7.508	9.083
230	172	58	6.371	33.026	14.745	16.892	28.048	27.159	7.509	9.085
240	179	61	6.371	33.027	14.746	16.894	28.052	27.159	7.510	9.086
250	187	63	6.372	33.029	14.748	16.895	28.056	27.160	7.511	9.088
260	194	66	6.373	33.030	14.749	16.896	28.059	27.160	7.512	9.089
270	202	68	6.373	33.032	14.751	16.898	28.062	27.160	7.513	9.090
280	209	71	6.374	33.033	14.752	16.899	28.065	27.160	7.514	9.091
290	217	73	6.374	33.034	14.753	16.900	28.068	27.161	7.515	9.092
300	224	76	6.375	33.035	14.754	16.900	28.070	27.161	7.515	9.093
310	232	78	6.375	33.036	14.755	16.901	28.073	27.161	7.516	9.094
320	239	81	6.376	33.037	14.756	16.902	28.074	27.161	7.517	9.094
330	247	83	6.376	33.038	14.757	16.903	28.077	27.162	7.517	9.095
340	254	86	6.377	33.039	14.758	16.904	28.079	27.162	7.518	9.096

Table 5b
Eigenvalues λ^2 for various modes (m, n): boundary conditions clamped–guided

N_1+N_2	N_1	N_2	$\alpha=\pi/2$					
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
50	12	38	1.898	19.455	4.853	8.366	13.309	13.406
60	14	46	1.918	19.666	4.910	8.454	13.309	13.565
70	17	53	1.936	19.861	4.959	8.535	13.310	13.704
80	19	61	1.947	19.984	4.990	8.586	13.310	13.791
90	22	68	1.958	20.101	5.021	8.634	13.310	13.876
100	24	76	1.965	20.179	5.041	8.666	13.311	13.933
110	27	83	1.972	20.259	5.061	8.699	13.311	13.990
120	29	91	1.977	20.314	5.075	8.721	13.312	14.029
130	32	98	1.982	20.371	5.089	8.745	13.312	14.070
140	34	106	1.986	20.411	5.099	8.761	13.312	14.099
150	37	113	1.990	20.454	5.110	8.778	13.313	14.129
160	39	121	1.993	20.485	5.118	8.791	13.313	14.151
170	42	128	1.996	20.518	5.127	8.805	13.313	14.175

Table 5b (*continued*)

N_1+N_2	N_1	N_2	$\alpha=\pi/2$					
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
180	44	136	1.998	20.543	5.133	8.814	13.314	14.193
190	47	143	2.000	20.570	5.139	8.825	13.314	14.212
200	49	151	2.002	20.590	5.144	8.833	13.314	14.226
210	52	158	2.004	20.611	5.150	8.842	13.314	14.242
220	54	166	2.006	20.628	5.154	8.849	13.314	14.253
230	57	173	2.007	20.646	5.159	8.856	13.315	14.266
240	59	181	2.009	20.660	5.162	8.862	13.315	14.276
250	62	188	2.010	20.676	5.166	8.868	13.315	14.287
260	64	196	2.011	20.687	5.169	8.873	13.315	14.296
270	67	203	2.012	20.701	5.173	8.878	13.315	14.305
280	69	211	2.013	20.711	5.175	8.882	13.315	14.312
290	72	218	2.014	20.723	5.178	8.887	13.315	14.321

Table 5c
Eigenvalues λ^2 for various modes (m,n): boundary conditions supported–free

N_1+N_2	N_1	N_2	$\alpha=\pi/2$					
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
50	12	38	0.571	15.803	2.801	5.428	10.450	8.132
60	14	46	0.572	15.802	2.803	5.429	10.452	8.133
70	17	53	0.574	15.815	2.812	5.434	10.472	8.137
80	19	61	0.575	15.815	2.813	5.435	10.473	8.138
90	22	68	0.577	15.827	2.816	5.439	10.481	8.140
100	24	76	0.577	15.828	2.816	5.439	10.482	8.140
110	27	83	0.578	15.833	2.819	5.442	10.489	8.142
120	29	91	0.578	15.833	2.820	5.442	10.489	8.142
130	32	98	0.578	15.839	2.821	5.444	10.493	8.143
140	34	106	0.578	15.839	2.821	5.444	10.493	8.143
150	37	113	0.579	15.842	2.823	5.445	10.497	8.144
160	39	121	0.579	15.842	2.823	5.445	10.497	8.144
170	42	128	0.579	15.845	2.824	5.446	10.499	8.144
180	44	136	0.579	15.845	2.824	5.446	10.499	8.144
190	47	143	0.579	15.847	2.824	5.447	10.501	8.145
200	49	151	0.580	15.847	2.825	5.447	10.502	8.145
210	52	158	0.580	15.849	2.825	5.448	10.503	8.145
220	54	166	0.580	15.849	2.825	5.448	10.503	8.145
230	57	173	0.580	15.850	2.826	5.448	10.505	8.145
240	59	181	0.580	15.850	2.826	5.448	10.505	8.146
250	62	188	0.580	15.852	2.826	5.449	10.506	8.146
260	64	196	0.580	15.852	2.826	5.449	10.506	8.146
270	67	203	0.580	15.853	2.827	5.449	10.507	8.146
280	69	211	0.580	15.853	2.827	5.449	10.507	8.146

frequency is less than 0.1%. Therefore, the accuracy of the natural frequency is definitely sufficient for engineering purposes. In the presented Tables 5(a)–(f)we notice the magnitude changes for λ^2 (proportional to the natural frequency ω) as small for all the numerical cases treated here. Table 5a exhibits such facts also for $m=0$, $n=1$ at $\alpha=\pi$, meaning half of the plate is clamped and the other half simply supported, as well as for

Table 5d
Eigenvalues λ^2 for various modes (m, n): boundary conditions free–guided

$N_1 + N_2$	N_1	N_2	$\alpha = \pi/2$				
			(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
50	12	38	9.270	2.836	2.954	8.456	7.965
60	14	46	9.276	2.837	2.961	8.459	7.998
70	17	53	9.282	2.837	2.967	8.461	8.027
80	19	61	9.285	2.837	2.971	8.462	8.045
90	22	68	9.289	2.838	2.974	8.464	8.060
100	24	76	9.291	2.838	2.977	8.465	8.071
110	27	83	9.293	2.838	2.979	8.465	8.082
120	29	91	9.295	2.838	2.980	8.466	8.089
130	32	98	9.296	2.838	2.982	8.466	8.096
140	34	106	9.297	2.838	2.983	8.467	8.101
150	37	113	9.298	2.839	2.984	8.467	8.107
160	39	121	9.299	2.839	2.985	8.467	8.111
170	42	128	9.300	2.839	2.986	8.468	8.115
180	44	136	9.301	2.839	2.986	8.468	8.118
190	47	143	9.301	2.839	2.987	8.468	8.121
200	49	151	9.302	2.839	2.987	8.468	8.124
210	52	158	9.302	2.839	2.988	8.468	8.126
220	54	166	9.303	2.839	2.988	8.468	8.129
230	57	173	9.303	2.839	2.989	8.469	8.131
240	59	181	9.304	2.839	2.989	8.469	8.133
250	62	188	9.304	2.839	2.989	8.469	8.134
260	64	196	9.304	2.839	2.990	8.469	8.136
270	67	203	9.305	2.839	2.990	8.469	8.137
280	69	211	9.305	2.839	2.990	8.469	8.139
290	72	218	9.305	2.839	2.990	8.469	8.140
300	74	226	9.305	2.839	2.991	8.469	8.141
310	77	233	9.306	2.839	2.991	8.469	8.142

Table 5e
Eigenvalues λ^2 for various modes (m, n): boundary conditions clamped–free

$N_1 + N_2$	N_1	N_2	$\alpha = \pi/2$					
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
50	12	38	1.175	16.946	3.153	6.530	11.211	9.343
60	14	46	1.169	16.927	3.144	6.514	11.195	9.342
70	17	53	1.165	16.926	3.140	6.506	11.194	9.344
80	19	61	1.162	16.912	3.134	6.496	11.183	9.344
90	22	68	1.160	16.911	3.131	6.490	11.177	9.346
100	24	76	1.157	16.900	3.127	6.483	11.169	9.346
110	27	83	1.156	16.896	3.124	6.478	11.165	9.347
120	29	91	1.154	16.887	3.121	6.473	11.158	9.347
130	32	98	1.153	16.884	3.119	6.469	11.154	9.348
140	34	106	1.152	16.877	3.116	6.465	11.148	9.348
150	37	113	1.151	16.874	3.115	6.462	11.145	9.348
160	39	121	1.150	16.868	3.113	6.459	11.141	9.348
170	42	128	1.150	16.865	3.111	6.457	11.138	9.349
180	44	136	1.149	16.860	3.110	6.454	11.134	9.349
190	47	143	1.149	16.857	3.109	6.452	11.132	9.349
200	49	151	1.148	16.854	3.107	6.450	11.129	9.349

Table 5e (*continued*)

N_1+N_2	N_1	N_2	$\alpha=\pi/2$					
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
210	52	158	1.148	16.851	3.106	6.448	11.127	9.350
220	54	166	1.147	16.848	3.105	6.446	11.124	9.350
230	57	173	1.147	16.845	3.104	6.445	11.122	9.350
240	59	181	1.147	16.842	3.103	6.443	11.120	9.350
250	62	188	1.146	16.840	3.103	6.442	11.118	9.350
260	64	196	1.146	16.838	3.102	6.441	11.116	9.350
270	67	203	1.146	16.836	3.101	6.439	11.115	9.350
280	69	211	1.146	16.834	3.100	6.438	11.113	9.350
290	72	218	1.145	16.832	3.100	6.437	11.112	9.351
300	74	226	1.145	16.830	3.099	6.436	11.110	9.351

Table 5f

Eigenvalues λ^2 for various modes (m, n): boundary conditions simply supported–guided

N_1+N_2	N_1	N_2	$\alpha=\pi/2$					
			(0,1)	(0,2)	(1,1) ①	(1,1) ②	(2,1) ①	(2,1) ②
50	12	38	1.748	19.686	4.934	7.884	13.637	11.917
60	14	46	1.777	19.761	4.965	7.958	13.707	11.923
70	17	53	1.807	19.860	5.013	8.042	13.823	11.929
80	19	61	1.824	19.915	5.032	8.090	13.872	11.933
90	22	68	1.845	20.000	5.056	8.156	13.933	11.937
100	24	76	1.856	20.043	5.069	8.190	13.967	11.940
110	27	83	1.868	20.095	5.088	8.229	14.018	11.943
120	29	91	1.876	20.126	5.097	8.253	14.045	11.945
130	32	98	1.885	20.172	5.109	8.286	14.077	11.947
140	34	106	1.891	20.198	5.116	8.305	14.097	11.948
150	37	113	1.897	20.229	5.126	8.327	14.126	11.950
160	39	121	1.901	20.249	5.132	8.342	14.142	11.951
170	42	128	1.907	20.278	5.139	8.362	14.162	11.953
180	44	136	1.910	20.295	5.144	8.374	14.175	11.954
190	47	143	1.914	20.315	5.150	8.388	14.193	11.955
200	49	151	1.917	20.329	5.154	8.398	14.204	11.956
210	52	158	1.920	20.348	5.158	8.411	14.217	11.956
220	54	166	1.923	20.361	5.161	8.419	14.226	11.957
230	57	173	1.925	20.375	5.165	8.429	14.239	11.958
240	59	181	1.927	20.386	5.168	8.436	14.246	11.959
250	62	188	1.930	20.399	5.171	8.445	14.256	11.959
260	64	196	1.931	20.408	5.174	8.452	14.263	11.960
270	67	203	1.933	20.419	5.177	8.459	14.272	11.960
280	69	211	1.934	20.427	5.179	8.464	14.278	11.961
290	72	218	1.936	20.438	5.181	8.471	14.285	11.961
300	74	226	1.937	20.445	5.183	8.476	14.290	11.962

$\alpha = 3\pi/2$. For $\alpha = \pi$ the axisymmetric natural frequency $m = 0$, $n = 1$ exhibits for $N_1 + N_2 = 340$ only an increase of 0.29%, while for $\alpha = 3\pi/2$ it is 0.3%. For the higher modes we have restricted the investigation to the case $\alpha = \pi/2$ and found for

clamped–simply supported: $m = 0$, $n = 2$: 0.12%, $m = 1$, $n = 1$ ①: 0.25%, $m = 1$, $n = 1$ ②: 0.19%, $m = 2$, $n = 1$ ①: 0.28%, $m = 2$, $n = 1$ ②: 0.03%

clamped–guided: $m = 0$, $n = 1$: 1.24%, $m = 0$, $n = 2$: 1.50%; $m = 1$, $n = 1$ ①: 1.50% $m = 1$, $n = 1$ ②: 1.45%, $m = 2$, $n = 1$ ①: 0.03%, $m = 2$, $n = 1$ ①: 1.54%
simply supported–free: $m = 0$, $n = 1$: 0.25%, $m = 0$, $n = 2$: 0.09%, $m = 1$, $n = 1$ ①: 0.15%, $m = 1$, $n = 1$ ②: 0.01%, $m = 2$, $n = 1$ ①: 0.03%, $m = 2$, $n = 1$ ②: 0.12%
free–guided: $m = 0$, $n = 2$: 0.25%, $m = 1$, $n = 1$ ①: 0.03%, $m = 1$, $n = 1$ ②: 0.23% $m = 2$, $n = 1$ ①: 0.03%, $m = 2$, $n = 1$ ②: 0.4%.

It may be mentioned that we have only looked at the convergence behavior of the method to the natural frequencies at $\alpha = \pi/2$, where we noticed acceptable results. For other α-values acceptable natural frequencies may need more or less points to satisfy our needs. We stopped adding additional points when an engineering acceptable result was reached. In Table 5a we also showed for this particular mixed boundary condition results for $\alpha = \pi$ and $\alpha = 3\pi/2$, which exhibited acceptable and very small deviations. Increasing the number $N_1 + N_2$ to even larger values, thus increasing the set of algebraic equations to a very high number, resulted in small oscillatory variations, indicating numerical instabilities.

We may conclude from these results that the increasing of the points at which the boundary conditions are satisfied does not require a large and numerically involved number $N_1 + N_2$ and that the results obtained with $N_1 + N_2 = 150$ warranty an acceptable convergence and the expected engineering accuracy for the natural frequencies.

It should be mentioned that the values of the pure boundary cases should go to the indicated points in spite of the fact that the proposed method is not capable to detach the plate completely at these locations, i.e. the plate is always connected with one point there. This would mean that the natural frequency would run for physical reasons (- - -) into these points of pure boundary conditions.

References

[1] W. Nowacki, Z. Olesiak, Vibration, buckling, and bending of a circular plate clamped along part of its periphery and simply supported on the remaining part, *Bulletin de L'Academie Polonaise des Sciences cl. IV* 4 (4) (1956) 247–258.

[2] C. Bartlett, B.A. Noble, A variational method for the solution of eigenvalue problems involving mixed boundary condition, *Applied Scientific Research B* 9 (1962) 403–419.

[3] C. Bartlett, The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder, *Quarterly Journal Mechanics and Applied Mathematics* 16 (1963) 431–440.

[4] B. Noble, The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder, *Proceedings of the Ninth Midwestern Conference on Solid and Fluid Mechanics*, 1965, pp. 141–146.

[5] H.F. Bauer, W. Eidel, Approximate natural frequencies of circular plates with mixed boundary conditions, Forschungsbericht, Universität der Bundeswehr München, LRT-WE-9-FB-1, 2004.

[6] A.W. Leissa, Vibration of plates, NASA-SP-160, US Government Printing Office, 1969, Washington, DC.